Asymptotic Behavior of Form Factors of Composite Particles

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Form factors for composite nucleons are studied in the large-momentum-transfer limit. It is argued that multiparticle intermediate states in the dispersion relation, containing arbitrarily large numbers of particles, must be included even to begin to estimate the asymptotic behavior. In the simplest approximation, this physical idea can be implemented using the ladder approximation in the nucleon legs of the form factor. The resulting form factors behave essentially like t^{-2} both for spin-0 and spin-$\frac{1}{2}$ nucleons.

I. INTRODUCTION

IT now seems to be well established experimentally that the nucleon electromagnetic form factors, at large spacelike momentum transfer t, are decreasing at least as fast as t^{-2}.[1]

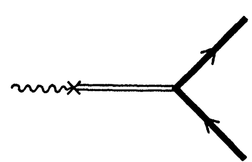
This fact has, however, caused a certain amount of consternation in some theoretical quarters, because it is difficult to reconcile with the "pole model" of form factors, which attempts to express the form factor as a sum of contributions from narrow $J^P=1^-$ resonances in the direct channel, as illustrated in Fig. 1(a). This is not to say that it is impossible to fit the experimental results with the sum of a few poles,[2] but such a fit does require certain miraculous and totally non-understood cancellations to occur between different poles, and it leaves many people unsatisfied.

On the other hand, one may well ask why anyone should expect the pole model to have anything at all to do with the large t behavior of form factors. Poles—that is, narrow resonances—represent structure in the direct, or t, channel. (In the simplest approximation, for example, as shown in Fig. 1(b), resonances in the t channel can be generated from "ladder graphs" in that channel.) No one would deny that the pole model is plausible for values of t near the mass squared of the resonance. But in other processes, such as two-body scattering, for example, high-t behavior is related to structure in the cross channel. As $t\rightarrow\infty$, one expects Regge behavior to result, and this, in the simplest approximation, is generated by ladder graphs in the cross channel, *not* in the direct channel.

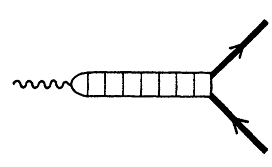
Any structure in the cross channel—in particular, the simple ladder graphs—gives rise to intermediate states in the direct channel containing arbitrarily large numbers of particles. In the case of the form factor, then, we should hardly expect to be able to say anything reasonable at all about the high-t behavior on the basis of models based on one- or two-particle cuts in the t channel.

We intend, in this paper, to present the simplest possible approximate calculation of the asymptotic behavior of a form factor which is consistent with the above reasoning. Our calculation will be very crude, but it will at least contain some contributions from states which, in the t channel, contain arbitrarily large numbers of particles. Since the approximations are drastic, we do not wish to take the detailed asymptotic t dependence too seriously, even though the behavior that we find is consistent with the experiments; however, we do wish to emphasize that whether or not one assumes the nucleon to be composite has a crucial impact on the degree of convergence: The form factor of a composite particle dies off at large t much faster than that of an elementary one. From this point of view, one can say that the fact that the nucleon form factors vanish at least as t^{-2} is evidence for the compositeness of the nucleon and, furthermore, that any falloff slower than t^{-2} would be very surprising if we believe that the nucleon is composite.

As we just stated, we do not necessarily believe the precise t dependence we obtain to be exactly correct. We shall therefore, in the interest of simplicity, confine



(a)



(b)

FIG. 1. (a) The type of diagrams encompassed by the "pole model" of form factors. The double line represents a vector meson, the wavy line is the photon, and the heavy solid line is a nucleon. (b) Classes of diagrams which can give rise to the pole model. Ladders of the sort shown can produce bound states or resonances like the vector meson in Fig. 1(a). The light solid lines are any particle, or particles, with quantum numbers such that the diagrams as drawn can exist.

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[1] D. H. Coward *et al.*, Phys. Rev. Letters 20, 292 (1968).

[2] V. Wataghin, Stanford University Report No. 272, 1967 (unpublished).

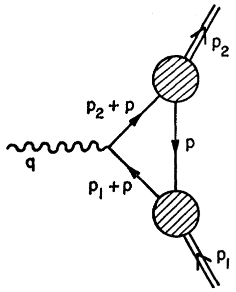


FIG. 2. The class of diagrams included in Eq. (2.1).

ourselves primarily to the case of zero spin. Thus the nucleon, the photon, and so forth, are all assumed to be spinless. The situation with the correct spins will be discussed briefly at the end.

Within our approximation, and in the spinless case, we find $F(t)\sim t^{-2}$ if the nucleon is composite, while $F(t)\sim \ln^2 t\, t^{-1}$ if it is elementary. More detailed approximations, such as inclusion of more structure in the cross channel, might increase the degree of convergence in the composite case (though we do not, of course, expect to find a faster falloff than e^{-t}).[3]

An outline of what follows is this: Section II contains a statement of our basic approximation for spinless particles, expressing the form factor in terms of the "triangle picture." Section III is devoted to potential theory. In Sec. IV we return to the relativistic theory and calculate the behavior of the nucleon vertex function; this result is applied in Sec. V to the asymptotic behavior of the form factor. Finally, in Sec. VI, we outline similar calculations for the physical spins and note the changes from the spinless case.

II. TRIANGLE APPROXIMATION

The simplest approximation to the nucleon electromagnetic form factor which incorporates the physical ideas discussed in the Introduction is illustrated in Fig. 2. The corresponding equation, for the case of spinless "nucleons" and spinless "photons" is

$$F(t)=i\int\frac{d^4p}{(2\pi)^4}\frac{1}{p^2-\mu^2}\frac{\Gamma^*(M^2;p^2,(p+p_2)^2)}{(p+p_2)^2-\mu^2}\times\frac{\Gamma(M^2;p^2,(p+p_1)^2)}{(p+p_1)^2-\mu^2}. \tag{2.1}$$

Here $q=p_2-p_1$ and $t=q^2$. The nucleon is considered to be a composite particle of mass M, made up primarily of two other particles of mass μ, which we shall henceforth refer to as the "constituent particles." The vertex function $\Gamma(p_2^2;p^2,(p+p_2)^2)$ describes the vertex connecting a nucleon of momentum p_2 to its two constituent particles of momenta $-p$ and $p+p_2$, respectively.[4]

In the coordinate system where q is purely timelike, we have

$$q=(\sqrt{t},0),\quad p_2=(\tfrac{1}{2}\sqrt{t},\mathbf{p}_1),\quad p_1=(-\tfrac{1}{2}\sqrt{t},\,\mathbf{p}_1), \tag{2.2}$$

with $|\mathbf{p}_1|=(\frac{1}{4}t-M^2)^{1/2}$. Thus, as t becomes very large,

$$(p+p_2)^2\to p^2+M^2+(\sqrt{t})(p_0-\mathbf{p}\cdot\hat{p}_1)$$
and
$$(p+p_1)^2\to p^2+M^2-(\sqrt{t})(p_0+\mathbf{p}\cdot\hat{p}_1). \tag{2.3}$$

If it is legitimate to interchange the limit $t\to\infty$ with the integration in Eq. (2.1), then we are required to know the behavior of the vertex function Γ when one of the elementary particles is very virtual, with its virtual-mass behavior like $\sqrt{t}$. In addition, two of the three energy denominators in Eq. (2.1) behave like $\sqrt{t}$. Roughly speaking, then, we can say that asymptotically $F(t)$ behaves (apart from logarithmic factors) like t^{-1} times whatever convergence comes from the vertex functions Γ. If $\Gamma\sim$const as $t\to\infty$, then we arrive at $F(t)\sim 1/t$. If Γ vanishes as $t\to\infty$, $F(t)$ will converge correspondingly faster. We shall return to a discussion of these alternatives more thoroughly and with more care in Sec. IV.

We could, if we so desired, also include structure in the t channel by using the approximation shown in Fig. 3(a). This amounts to inserting under the integral in Eq. (2.1) some completely off-mass-shell form factor $G(t;(p_2+p)^2,(p_1+p)^2)$. Whatever the behavior of this function is, it can only make $F(t)$ converge more rapidly as $t\to\infty$ than it does if G is replaced by 1. For example, we can incorporate the usual pole model into our approximation by writing G as

$$G(t;(p_2+p)^2,(p_1+p)^2)=\sum_n\frac{\gamma_n}{t-m_n^2}, \tag{2.4}$$

corresponding to diagrams of the type shown in Fig. 3(b). For this choice of G, the form factor F converges by at least one power of t faster than the approximation of Eq. (2.1).

III. NONRELATIVISTIC LIMIT AND POTENTIAL THEORY

Let us now digress from the main discussion and look at the nonrelativistic limit of Eq. (2.1). By this we mean the limit as M and $\mu\to\infty$. It is then straightforward to evaluate the integral over dp_0 in Eq. (2.1). There are six poles in the integrand, two at

$$p_0=\pm(\mathbf{p}^2+\mu^2)^{1/2}\mp i\epsilon$$

and double poles at each of the points

$$p_0=-M\pm\mu\mp i\epsilon.$$

[3] A. Jaffe, Phys. Rev. Letters 17, 661 (1966).

[4] We use the notation Γ rather than F for this quantity to emphasize that in our approximations we shall not be able to include bubbles on the constituent particle legs. Since we are interested in evaluating Γ for the nucleon on the mass shell, bubbles on the nucleon leg do not contribute in any event.

If the contour is closed in the upper half-plane, only the pole at $p_0=-(\mathbf{p}^2+\mu^2)^{1/2}+i\epsilon$ contributes; the double poles cancel. One thus obtains from Eq. (2.1)

$$F(t)=\frac{1}{2\mu}\int\frac{d^3\mathbf{p}}{(2\pi)^3}\Gamma^*(M^2;M[E_B-(\mathbf{p}+\tfrac{1}{2}\mathbf{p}_2)^2/\mu]+M^2,M^2)$$
$$\times\frac{1}{M[E_B-(\mathbf{p}+\frac{1}{2}\mathbf{p}_2)^2/M]}\frac{1}{M[E_B-(\mathbf{p}+\frac{1}{2}\mathbf{p}_1)^2/M]}$$
$$\times\Gamma(M^2;M[E_B-(\mathbf{p}+\tfrac{1}{2}\mathbf{p}_1)^2/\mu]+M^2,M^2),\quad(3.1)$$

where

$$E_B=M-2\mu.\quad(3.2)$$

Equation (3.1) is the exact expression for the form factor in potential theory, with the definition

$$\frac{1}{(2\mu M^2)^{1/2}}\Gamma(M^2;M[E_B-(\mathbf{p}+\tfrac{1}{2}\mathbf{p}_1)^2/\mu]+M^2,M^2)$$
$$=\Gamma(\mathbf{p}+\tfrac{1}{2}\mathbf{p}_1)=\langle\mathbf{p}+\tfrac{1}{2}\mathbf{p}_1|V|\psi_B\rangle,\quad(3.3)$$

where ψ_B is the bound-state wave function. This fact is easily seen. We have

$$F(t)=\langle\psi_B|e^{i\mathbf{q}\cdot\mathbf{x}/2}|\psi_B\rangle,\quad(3.4)$$

where $t=-\mathbf{q}^2$.[5] But

$$|\psi_B\rangle=(E_B-H_0)^{-1}V|\psi_B\rangle,\quad(3.5)$$

so that

$$F(t)=\int\frac{d^3\mathbf{p}}{(2\pi)^3}$$
$$\times\frac{\langle\psi_B|V|\mathbf{p}+\frac{1}{2}\mathbf{p}_2\rangle\langle\mathbf{p}+\frac{1}{2}\mathbf{p}_1|V|\psi_B\rangle}{[E_B-(\mathbf{p}+\frac{1}{2}\mathbf{p}_2)^2/\mu][E_B-(\mathbf{p}+\frac{1}{2}\mathbf{p}_1)^2/\mu]},\quad(3.6)$$

which is identical with Eq. (3.1).

Our basic approximation, Eq. (2.1), may thus be thought of as the direct relativistic generalization of potential theory.

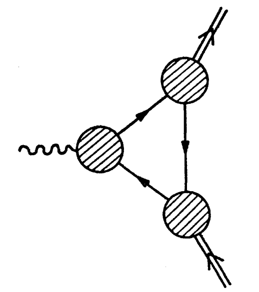
Let us pursue the potential theory analogy a bit further. We need to learn something about the function $\Gamma(\mathbf{q})$, and we may evidently write

$$\Gamma(\mathbf{q})=\left\langle\mathbf{q}\left|V\frac{1}{E_B-H_0}V\right|\psi_B\right\rangle.\quad(3.7)$$

Thus if there exists an unperturbed state ψ_{B_0}, satisfying

$$H_0\psi_{B_0}=E_{B_0}\psi_{B_0},\quad(3.8)$$

[5] It is more usual to see Eq. (3.4) without the factor of $\frac{1}{2}$ in the exponent. However, we are dealing with two nonrelativistic particles of mass μ interacting through a potential V, and not with one particle in some fixed central potential. Thus, the form factor, strictly, is $(2\pi)^3\delta^3(p'-\mathbf{p}-\mathbf{q})F(t)=\langle\psi_B,\mathbf{p}'|e^{i\mathbf{q}\cdot\mathbf{x}_1}|\psi_B,\mathbf{p}\rangle$, where $\mathbf{p}'$, $\mathbf{p}$ are the final and initial c.m. momenta, and $\mathbf{x}_1$ is the position of *one* of the particles. Once the c.m. motion is removed, Eq. (3.4) results. By the same token, in Eq. (3.6) and similar equations written later, the reduced mass $\mu^2/(\mu+\mu)=\frac{1}{2}\mu$ appears everywhere; thus we find expressions like $(\mathbf{p}+\frac{1}{2}\mathbf{p}_2)^2/\mu$ rather than $(\mathbf{p}+\mathbf{p}_2)^2/2\mu$ as one might at first glance expect.



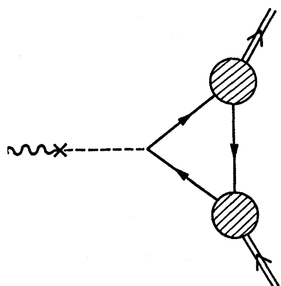


FIG. 3. (a) A more complicated class of diagrams which could also be used. The blob on the left-hand corner is supposed to include t-channel ladders such as that shown in Fig. 1(b). (b) Incorporation of the usual pole model into the approximation. The dashed line represents one of the particles in the sum in Eq. (2.3).

which becomes the state ψ_B after the perturbation V is turned on, that is, if ψ_B does not represent a true bound state created by the potential V but rather reflects the existence of an "elementary" particle in H_0, then we find

$$\Gamma(\mathbf{q})=(\sqrt{Z_B})\Gamma_0(\mathbf{q})+\int\frac{d^3\mathbf{p}}{(2\pi)^3}\frac{\langle\mathbf{q}|V|\mathbf{p}\rangle\Gamma(\mathbf{p})}{E_B-p^2/\mu},\quad(3.9)$$

where

$$\sqrt{Z_B}=\langle\psi_{B_0}|\psi_B\rangle$$

and

$$\Gamma_0(\mathbf{q})=\langle\mathbf{q}|V|\psi_{B_0}\rangle.\quad(3.10)$$

We can now compare, within the framework of potential theory, the vertex function Γ and hence the form factor F for the elementary and composite cases.

If an elementary particle exists, we expect $\psi_{B_0}(\mathbf{x})$ to contain a term proportional to $\delta^3(\mathbf{x})$, representing the existence of some elementary "core"; thus, $\Gamma_0(\mathbf{q})\to$ const as $\mathbf{q}^2\to\infty$, and hence $\Gamma(\mathbf{q})\to$ const as well. The behavior of the form factor may then be derived from Eq. (3.6). We find[6]

$$F(t)\to\text{const}\times\int d^3\mathbf{p}\frac{1}{[E_B-(\mathbf{p}+\frac{1}{2}\mathbf{p}_2)^2/\mu]}$$
$$\times\frac{1}{[E_B-)\mathbf{p}+\frac{1}{2}\mathbf{p}_1)^2/\mu]}\sim1/\sqrt{t}.\quad(3.11)$$

If, on the other hand, ψ_B does not represent an elementary particle, then ψ_{B_0} does not exist, $Z_B=0$, and

[6] This asymptotic behavior, namely, $t^{-1/2}$, is not the same as that obtained in the equivalent relativistic case, which is $(\ln t)^2/t$. This is because the limits $t\to\infty$ and $M,\mu\to\infty$ are not, in general, interchangeable.

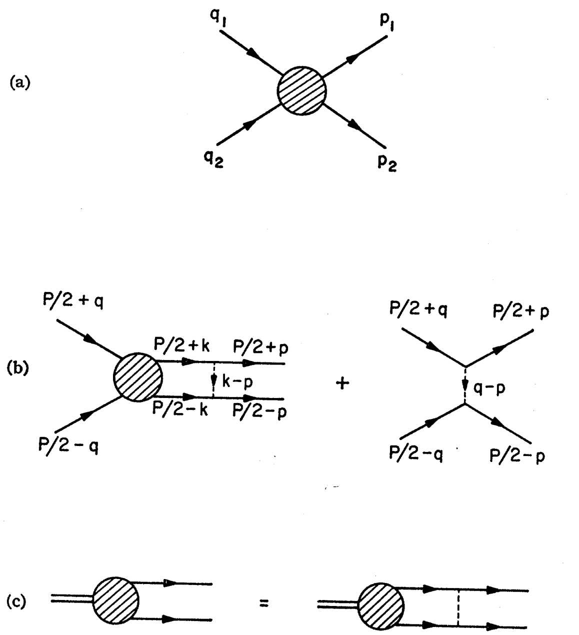


FIG. 4. (a) The "constituent particle" scattering process. (b) The ladder approximation to the constituent particle scattering amplitude. (c) The ladder approximation to the composite vertex function.

the inhomogeneous term disappears from Eq. (3.11). In this case, the asymptotic behavior of $\Gamma(\mathbf{q})$, for a Yukawa potential, for example, is

$$\Gamma(\mathbf{q})\sim 1/q^2$$

and that of $F(t)$ is

$$F(t)\sim 1/t^2.$$

The contrast is complete. In the composite case, $F(t)$ decreases by more than a full power of t faster than in the elementary case. As we shall see in Sec. V, a similar result follows, within our approximation, in the relativistic case.

There are a number of other relations involving the vertex $\Gamma(\mathbf{q})$ which are valid in potential theory and which have their analogs as approximations to the relativistic theory. Since we shall make use of some of these approximations later, it may be of some value to write them in the context of potential theory.

We may define the scattering matrix by

$$\begin{aligned}T(s)&=V+V(s-H_0+i\epsilon)^{-1}T(s)\\&=V+V(s-H+i\epsilon)^{-1}V.\end{aligned}\tag{3.12}$$

The off-shell scattering amplitude is now given by

$$\langle\mathbf{q}|T(s)|\mathbf{p}\rangle=\langle\mathbf{q}|V|\mathbf{p}\rangle+\frac{\Gamma(\mathbf{q})\Gamma^*(\mathbf{p})}{s-E_B}+\int\frac{d^3\mathbf{p}'}{(2\pi)^3}\frac{T(\mathbf{q},\mathbf{p}')T(\mathbf{p},\mathbf{p}')^*}{s-E_{p'}+i\epsilon},\tag{3.13}$$

where

$$T(\mathbf{q},\mathbf{p})=\langle\mathbf{q}|T(s)|\mathbf{p}\rangle|_{s=p^2/\mu}=\langle\mathbf{q}|V|\psi_{\mathbf{p}}^{(+)}\rangle.\tag{3.14}$$

Thus we have

$$(s-E_B)\langle\mathbf{q}|T(s)|\mathbf{p}\rangle|_{s=E_B}=\Gamma(\mathbf{q})\Gamma^*(\mathbf{p})\tag{3.15}$$

and

$$(s-E_B)T(\mathbf{q},\mathbf{p})|_{s=E_B}=g\Gamma(\mathbf{q}),\tag{3.16}$$

where

$$g=\Gamma(\mathbf{p})|_{p^2/\mu=E_B}.\tag{3.17}$$

IV. RELATIVISTIC CASE

We now return to the relativistic case. In order to make use of Eq. (2.1), we need to know something about the vertex function Γ, and to this end we turn to the off-mass-shell scattering amplitude for the two constituent particles of the nucleon. This amplitude is a function of six variables, and we write it as $T(s,t;q_1^2q_2^2p_1^2p_2^2)$, where $s=(q_1+q_2)^2=(p_1+p_2)^2$, and $t=(q_1-p_1)^2=(q_2-p_2)^2$. The scattering process described by this T is illustrated in Fig. 4(a); q_1 and q_2 are the momenta of the initial two particles and p_1 and p_2 of the final two particles. The physical scattering amplitude is simply

$$T(s,t)=T(s,t;\mu^2\mu^2\mu^2\mu^2).\tag{4.1}$$

The amplitude T has a pole in s at $s=M^2$, the nucleon mass, and as in potential theory we may write

$$\begin{aligned}&\Gamma(M^2;q_1^2,q_2^2)\Gamma(M^2;p_1^2,p_2^2)^*\\&\quad=(s-M^2)T(s,t;q_1^2,q_2^2,p_1^2,p_2^2)|_{s=M^2}.\end{aligned}\tag{4.2}$$

We shall assume, in order to have an explicit model, that the force between the two constituent particles of mass μ is due to their interaction with another spinless particle of mass m. We shall call the coupling constant to this particle f.

Let us now use the ladder approximation to calculate T. It is important to emphasize that use of the ladder approximation here in the s channel is entirely consistent with our earlier remarks about the necessity of including structure in the cross channel to evaluate $F(t)$ for large t. Here we are interested in the scattering amplitude for low values of s, in the vicinity of the nucleon mass. We have, as illustrated in Fig. 4(b), the equation

$$T(P;q,p)=f^2/[(p-q)^2-m^2]+\frac{if^2}{16\pi^4}\int d^4k\times\frac{T(P;k,p)}{[(p-k)^2-m^2][(\frac{1}{2}P+k)^2-\mu^2][(\frac{1}{2}P-k)^2-\mu^2]},\tag{4.3}$$

where we define

$$P=p_1+p_2=q_1+q_2,$$
$$p=\tfrac{1}{2}(p_1-p_2),\quad q=\tfrac{1}{2}(q_1-q_2),$$

and

$$T(P^2,(p-q)^2;\,(\tfrac{1}{2}P+q)^2,(\tfrac{1}{2}P-q)^2,(\tfrac{1}{2}P+p)^2,(\tfrac{1}{2}P-p)^2)\\ =T(P;\,q,p). \quad (4.4)$$

This approximation to T has the following virtues:

(i) The ladder approximation leads to Regge asymptotic behavior in t corresponding to a composite nucleon pole in the s channel.

(ii) It is the direct relativistic analog of potential theory.

(iii) It is closely related to the various bootstrap models that have been used extensively to describe composite particles.

We need to make no further approximations to obtain the asymptotic behavior of $\Gamma(M^2;\,q_1^2,q_2^2)$ for large q_1^2. Before doing this, however, we think it of value to digress briefly to the N/D approximation to Eq. (4.3), because this further approximation permits the distinction between the elementary and composite situations to be made very clear.

In this approach, one constructs a solution for the s-wave scattering amplitude T_0 which has the left-hand cut in the s plane given by the single particle exchange term and the right-hand cut given by the two-particle unitary cut, which can be calculated by replacing in Eq. (3.2) the propagators for the particles of mass μ by δ functions:

$$[(\tfrac{1}{2}P\mp k)^2-\mu^2]^{-1}\rightarrow 2\pi\delta((\tfrac{1}{2}P\mp k)^2-\mu^2)\theta(\tfrac{1}{2}P_0\mp k_0).$$

For $s_1=p_1^2$ and $s_2=p_2^2$ sufficiently small, and for $q_1^2=q_2^2=\mu^2$, we obtain

$$T_0(s+i\epsilon,\,s_1,s_2)-T_0(s-i\epsilon,,s_1,s_2)\\ =\frac{2i}{16\pi}\left(\frac{s-4\mu^2}{4s}\right)^{1/2}T_0(s,s_1,s_2)T_0^*(s) \quad (4.5)$$

for $s>4\mu^2$. Here we use the convention that quantities with only one argument are on-shell quantities, i.e., $T_0(s)=T_0(s,\mu^2,\mu^2)$ and, similarly,

$$T_0(s;\,s_1s_2)=T_0(s;\,\mu^2\mu^2s_1s_2).$$

The S-wave projection of the one-particle exchange diagram is given by

$$B(s,s_1,s_2)=\frac{f^2}{2qp}Q_0\left(\frac{2m^2+s-s_1-s_2-2\mu^2}{4qp}\right), \quad (4.6)$$

where

$$q^2=\tfrac{1}{4}s-\mu^2 \quad\text{and}\quad p^2=\tfrac{1}{4}s-\tfrac{1}{2}(s_1+s_2)+(s_1-s_2)^2/4s. \quad (4.7)$$

The function $T_0(s)$ can be written in the form

$$T_0(s)=N(s)/D(s), \quad (4.8)$$

where $N(s)$ is the solution of the integral equation

$$N(s)=B(s)+\frac{1}{16\pi^2}\int_{4\mu^2}^{\infty}ds'\left(\frac{s'-4\mu^2}{4s'}\right)^{1/2}\\ \times\frac{B(s')-B(s)}{s'-s}N(s') \quad (4.9)$$

and where

$$D(s)=1-\frac{1}{16\pi^2}\int_{4\mu^2}^{\infty}ds'\left(\frac{s'-4\mu^2}{4s'}\right)^{1/2}\frac{N(s')}{s'-s}. \quad (4.10)$$

In terms of these quantities, $T(s,s_1,s_2)$ can be calculated by quadrature:

$$T(s,s_1,s_2)=B(s,s_1,s_2)\\ -\frac{1}{D(s)\pi}\int_{4\mu^2}^{\infty}ds'\frac{B(s',s_1,s_2)\,\mathrm{Im}D(s')}{(s'-s)} \quad (4.11)$$

for s_1 and s_2 sufficiently small.

Finally, to obtain Γ we use the fact that

$$g\Gamma(M^2;\,s_1,s_2)=(s-M^2)T(s;\,s_1,s_2)\,|_{s=M^2}. \quad (4.12)$$

The distinction between the elementary and composite cases is now quite clear. If the nucleon is composite, the pole in T occurs through the vanishing of D, and we find

$$g\Gamma(M^2;\,s_1,s_2)\\ =-\left(\frac{1}{\pi}\int_{4\mu^2}^{\infty}ds'\frac{B(s';\,s_1,s_2)\,\mathrm{Im}D(s')}{s'-M^2}\right)\Big/D'(s)\,|_{s=M^2}. \quad (4.13)$$

If, on the other hand, the nucleon is elementary, the pole in T is due to a pole in the input B. Therefore, in this case,

$$g\Gamma(M^2;\,s_1,s_2)=(s-M^2)B(s;\,s_1,s_2)\,|_{s=M^2}, \quad (4.14)$$

and at our level of approximation this amounts simply to saying

$$\Gamma(M^2;\,s_1,s_2)=\Gamma(M^2;\,\mu^2,\mu^2)=g. \quad (4.15)$$

If the nucleon is elementary, then we have $\Gamma\rightarrow$ const as $s_1\rightarrow\infty$.

In the composite case, the asymptotic behavior of Γ may be obtained from Eq. (4.13). As s_1 is increased, the left-hand branch point in the s plane of the one-particle exchange term moves through the unitarity cut and the contour must be deformed around the singularity of B, thus producing an anomalous threshold. The resulting expression is

$$\Gamma(M^2;s_2,s_3)=-\frac{1}{D'(M^2)}\left\{\frac{1}{16\pi^2}\int_{s_-(s_1)}^{4\mu^2}\frac{\operatorname{disc}[B(s';s_1,s_2)]}{s'-M^2}N(s')\left(\frac{s'-4\mu^2}{s'}\right)^{1/2}+\frac{1}{\pi}\int_{4\mu^2}^{\infty}ds'\frac{B(s';s_1,s_2)\operatorname{Im}D(s')}{(s'-M^2)}\right\},\quad(4.16)$$

where, for large s_1,

$$s_-(s_1)=\frac{s_1}{2M^2}\{M^2+\mu^2-s_2-[(M^2+\mu^2-s_2)^2-4M^2\mu^2]^{1/2}\}.$$

The discontinuity of B across the anomalous cut vanishes more rapidly as a function of s and s_1 than B itself, leaving the normal integral dominating the asymptotic behavior of Γ.

The asymptotic behavior of $B(s)$, as $s\to\infty$, is $(\ln s)/s$. Therefore, $N(s)$ vanishes like s^{-1} (apart from factors of $\ln s$) as $s\to\infty$, and hence so also does $\operatorname{Im}D(s)$.

The asymptotic behavior of $B(s;s_1s_2)$, as $s_1\to\infty$, is

$$B(s;s_1,s_2)\to\frac{1}{s_1}\left(\frac{s}{s-4\mu^2}\right)^{1/2}f^2\ln\frac{s+(s-4\mu^2)^{1/2}}{s-(s-4\mu^2)^{1/2}}.\quad(4.17)$$

Hence the limit $s_1\to\infty$ may be interchanged with the integral over ds' in Eq. (4.16), and we find that $\Gamma(M^2;s_1,s_2)$ behaves like const$/s_1$ as $s_1\to\infty$.

Let us now leave the N/D digression and return to the full ladder approximation, Eq. (4.3). From Eq. (4.3), and using Eq. (4.2), we obtain for the composite nucleon case the result

$$\Gamma(M^2;(\tfrac{1}{2}P+q)^2,(\tfrac{1}{2}P-q)^2)=\frac{if^2}{16\pi^4}\int d^4k\frac{\Gamma(M^2;(\frac{1}{2}P+k)^2,(\frac{1}{2}P-k)^2)}{[(k-q)^2-m^2][(\frac{1}{2}P+k)^2-\mu^2][(\frac{1}{2}P-k)^2-\mu^2]},\quad(4.18)$$

which corresponds to the diagram shown in Fig. 4(c).

If the nucleon were elementary, there should have been a direct channel nucleon pole in the inhomogeneous term in Eq. (4.3), and we would again obtain simply $\Gamma=g$ as our result.

Now in the c.m. system, $P=(M,0)$ so that if $s_1=(\frac{1}{2}P+q)^2$ and $s_2=(\frac{1}{2}P-q)^2$, then

$$q^2=\tfrac{1}{4}M^2+\tfrac{1}{2}(s_1+s_2),\quad q_0=(s_1-s_2)/2M,$$

and

$$|\mathbf{q}|^2=[M^4-2M^2(s_1+s_2)+(s_1-s_2)^2]/4M^2.\quad(4.19)$$

The only dependence on s_1 in the integrand of Eq. (4.18) is through the factor $(k-q)^2$; for large s_1, this factor approaches

$$(k-q)^2\to s_1\left(\frac{1}{2}-\frac{k_0}{M}+\frac{\mathbf{k}\cdot\hat{q}}{M}\right).\quad(4.20)$$

Thus the limit $s_1\to\infty$ may be interchanged with the integral in Eq. (4.18), and we finally obtain the same behavior that we found in the N/D approximation:

$$\Gamma(M^2;s_1,s_2)\xrightarrow[s_1\to\infty]{}C/s_1,$$

where

$$C=\frac{if^2}{16\pi^4}\int d^4k\frac{\Gamma(M^2;(\frac{1}{2}P+k)^2,(\frac{1}{2}P-k)^2)}{(\frac{1}{2}-k_0/M+\mathbf{k}\cdot\hat{q}/M)[(\frac{1}{2}P+k)^2-\mu^2][(\frac{1}{2}P-k)^2-\mu^2]}.\quad(4.21)$$

V. ASYMPTOTIC BEHAVIOR OF THE FORM FACTOR

We now return to Eq. (2.1), armed with our knowledge of the behavior of Γ, at least in the ladder approximation. The limit $t\to\infty$ can be interchanged with the integral over d^4p, in the composite nucleon case, because if this is done, then as we saw in Eq. (2.2), the virtual mass squareds behave like $(\sqrt{t})p$ so that the integral over d^4p still exists. The result is that F behaves like t^{-2}, just as in the potential theory case, as $t\to\infty$.

The process may be made a bit more explicit as follows. We found

$$\Gamma(M^2;s_1,s_2)\to C/s_1$$

as $s_1\to\infty$. Substituting this into the integral in Eq.

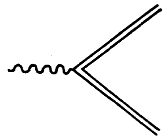


FIG. 5. Contact interaction of a photon and an elementary nucleon.

(2.1), assuming that only large values of $(p+p_2)^2$ and $(p+p_3)^2$ are important, we find

$$F(t)\underset{t\to\infty}{\longrightarrow} i\int\frac{d^4p}{(2\pi)^4}\frac{1}{p^2-\mu^2}\frac{|C|^2}{(p+p_2)^4(p+p_3)^4}. \tag{5.1}$$

The integral here can be evaluated explicitly, and we find

$$F(t)\longrightarrow\frac{1}{16\pi^2\mu^2}\frac{|C|^2}{t^2}. \tag{5.2}$$

If a better evaluation of the behavior of Γ for large virtual masses shows a faster falloff, such as, for example, exponential, then F will converge correspondingly faster.[7]

The elementary nucleon case is quite different. As we saw in Sec. IV, in our approximation at least (and one would expect this to be true generally) Γ approaches a constant at large virtual masses. In this case we expect, if

$$\Gamma(M^2;\,s_1,s_2)\underset{s_1\to\infty}{\longrightarrow}\gamma,$$

then from Eq. (2.1)

$$\begin{aligned}F(t)\longrightarrow i|\gamma|^2\int\frac{d^4p}{(2\pi)^4}\frac{1}{p^2-\mu^2}\frac{1}{(p+p_2)^2-\mu^2}\frac{1}{(p+p_3)^2-\mu^2}\\ =\frac{|\gamma|^2}{16\pi^2}\frac{[\ln(t/\mu^2)]^2}{t}.\end{aligned} \tag{5.3}$$

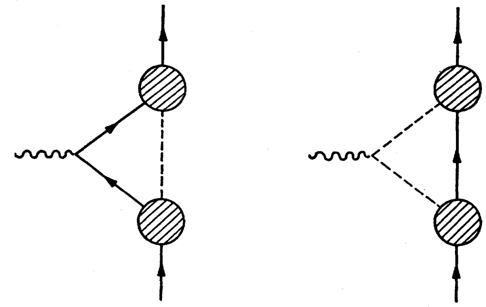


FIG. 6. Diagrams for the form factors of a spin-$\frac{1}{2}$ nucleon. The solid line is a nucleon and the dashed line a pion.

On the other hand, if the nucleon is elementary, there is presumably also the additional contact diagram illustrated in Fig. 5, which is not included in Eq. (2.1). If we include this as well, we would expect that in the elementary case,

$$F(t)\longrightarrow \text{const}+\frac{|\gamma|^2}{16\pi^2}\frac{[\ln(t/\mu^2)]^2}{t}. \tag{5.4}$$

In any event, whether or not the contact diagram exists and is included, the elementary nucleon form factor is far less convergent at large t than is the composite nucleon form factor.

VI. CASE WITH SPIN

The case of spin-$\frac{1}{2}$ nucleons can be treated in the same way, and the results are similar. The analog of Eq. (2.1) is

$$\begin{aligned}&F_1(t)\gamma_\mu+iF_2(t)\sigma_{\mu\nu}q_\nu\\ &=i\int\frac{d^4p}{(2\pi)^4}\frac{1}{p^2-\mu^2}\frac{1}{(p+p_1)^2-M^2}\frac{1}{(p+p_2)^2-M^2}[\Gamma_1{}^*(M^2,(p+p_2)^2;\,p^2)\gamma_5+\Gamma_2{}^*(M^2,(p+p_2)^2;\,p^2)\gamma_5(\not p+\not p_2-M)]\\ &\quad\times(\not p+\not p_2+M)\gamma_\mu(\not p+\not p_1+M)[\Gamma_1(M^2,(p+p_1)^2;\,p^2)\gamma_5+\Gamma_2(M^2,(p+p_1)^2;\,p^2)(\not p+\not p_1-M)\gamma_5]\\ &\quad+i\int\frac{d^4p}{(2\pi)^4}\frac{1}{p^2-M^2}\frac{1}{(p+p_2)^2-\mu^2}\frac{1}{(p+p_1)^2-\mu^2}[\Gamma_1{}^*(M^2,p^2;\,(p+p_2)^2)\gamma_5+\Gamma_2{}^*(M^2,p^2;\,(p+p_2)^2)\gamma_5(\not p-M)]\\ &\quad\times(\not p+M)(2p_\mu+p_{2\mu}+p_{1\mu})[\Gamma_1(M^2,p^2;\,(p+p_1)^2)\gamma_5+\Gamma_2(M^2,p^2;\,(p+p_1)^2)(\not p-M)\gamma_5],\end{aligned} \tag{6.1}$$

and corresponds to the diagrams in Fig. 6. In Eq. (6.1), we have written the pion-nucleon vertex function, for an off-mass-shell pion of momentum q and an off-mass-shell nucleon of momentum p, as

$$[\Gamma_1(M^2,p^2;\,q^2)\gamma_5+\Gamma_2(M^2,p^2;\,q^2)(\not p-M)\gamma_5], \tag{6.2}$$

and F_1 and F_2 are the usual electromagnetic form factors.

As before, let us assume the nucleon to be composite and to appear as a bound state in the $p_{1/2}$ channel of pion-nucleon scattering. Further, let us suppose that the binding force between the pion and nucleon arises from the exchange of some scalar meson of mass m, which couples to pions with a coupling constant f_π and to nucleons with a coupling constant f_N. We do not mean to imply that we believe this model to be in any way a realistic description

[7] J. Harte [Phys. Rev. **165**, 1557 (1968)] has discussed more sophisticated approximations to Γ than the ladder approximation and argues that Γ may in fact decrease like $\exp(-\sqrt{s_1})$.

of the interaction between pions and nucleons; indeed, our results do not depend on any particular model. Nevertheless, a model serves to make our approximations explicit, and the one described above is chosen for simplicity.

Within this model, we may write the analog of Eq. (4.18) for the pion-nucleon vertex in the ladder approximation.

$$\Gamma_1(M^2,(\tfrac{1}{2}P+q)^2;\ (\tfrac{1}{2}P-q)^2)\gamma_5+\Gamma_2(M^2,(\tfrac{1}{2}P+q)^2;\ (\tfrac{1}{2}P-q)^2)(\tfrac{1}{2}\boldsymbol{P}+\boldsymbol{q}-M)\gamma_5=\frac{if_Nf_\pi}{16\pi^4}\int d^4k\frac{1}{(k-q)^2-m^2}\frac{1}{(\tfrac{1}{2}P+k)^2-M^2}$$
$$\times\frac{1}{(\tfrac{1}{2}P-k)^2-\mu^2}(\tfrac{1}{2}\boldsymbol{P}+\boldsymbol{k}+M)[\Gamma_1(M^2,(\tfrac{1}{2}P+k)^2;\ (\tfrac{1}{2}P-k)^2)\gamma_5+\Gamma_2(M^2,(\tfrac{1}{2}P+k)^2;\ (\tfrac{1}{2}P-k)^2)(\tfrac{1}{2}\boldsymbol{P}+\boldsymbol{k}-M)\gamma_5]. \quad (6.3)$$

The spin algebra here is easily worked out, and the limit as $s_1=(\frac{1}{2}P+q)^2$ becomes large can then be taken. We find

$$\Gamma_1(M^2,s_1;s_2)\to C_1/s_1 \quad\text{and}\quad \Gamma_2(M^2,s_1;s_2)\to C_2/s_1^2 \quad (6.4)$$

as $s_1\to\infty$. The same asymptotic behavior in s_2 holds if we let $s_2\to\infty$.

We can now go back to Eq. (6.1). After the spin algebra is calculated and separate equations for F_1 and F_2 are obtained, the asymptotic behavior resulting from Eqs. (6.4) comes out to be

$$F_1(t)\sim(\ln t)^2/t^2 \quad\text{and}\quad F_2(t)\sim 1/t^2.$$

Thus essentially the same conclusions obtain here as in the spinless case.

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Study of Bound-State Solutions to the Pion-Nucleon Bethe-Salpeter Equation*

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The Bethe-Salpeter equation describing the interaction of pseudoscalar mesons and nucleons via pseudoscalar coupling is solved numerically for energies below the elastic threshold by use of variational techniques. We consider only the "ladder" approximation with a local potential corresponding to the exchange of an elementary nucleon. Simple generalizations of this form of the interaction are considered as well. In the absence of a cutoff, this leads to a marginally singular integral equation. We examine in detail the boundary conditions to be imposed on the solutions in order to lead to a discrete eigenvalue spectrum. The study of this problem is considerably simplified at zero total c.m. energy, where the (Wick-rotated) equation is invariant under four-dimensional rotations. In order to take full advantage of this symmetry, we construct a new set of spinor spherical harmonics belonging to the representations $(\frac{1}{2}(n\pm1),\frac{1}{2}n)$ and $(\frac{1}{2}n,\frac{1}{2}(n\pm1))$ of the four-dimensional rotation group. The discussion is then extended to the general case, in which we examine briefly the formal structure of the $E\neq0$ solutions.

I. INTRODUCTION

In recent years there has been renewed interest in the relativistic two-body equations of Salpeter and Bethe.[1] In the absence of a theory of the strong interactions, these off-shell equations provide at least a means for performing dynamical calculations within a manifestly covariant framework. However, even in the "ladder" approximation, in which we retain only the lowest-order term in the expansion of the interaction in powers of G^2 (the square of the coupling constant), the equation has for some time been considered intractable, the difficulties being largely due to the presence of a degree of freedom in the equation, the "relative time," which has no analog in nonrelativistic quantum mechanics. The numerical program initiated by Schwartz[2] demonstrated, however, that the (Wick-rotated) Bethe-Salpeter (BS) equation, in the ladder approximation, could be solved accurately by conventional numerical techniques. This led to renewed interest in the BS equation as a computational tool, a number of calculations having extended since then the bound-state calculation by Schwartz to the elastic scattering region[3] and as far as the second inelastic threshold.[4]

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